COUPLED FIXED POINT THEOREMS FOR MAPPINGS SATISFIES CONTRACTIVE CONDITION IN CONE METRIC SPACE

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ABSTRACT In the past few years, there are several authors have come up with coupled fixed point theorems in cone metric space. The purpose of this paper to prove the existence of a coupled fixed point of some type of contraction mappings defined on a complete cone metric space. It extends and generalizes many previous coupled fixed point theorems.

Keywords: Metric space, cone metric space, coupled fixed point, mixed monotone property

1. INTRODUCTION Fixed point theory is a well-known and significant area in mathematics, with a wide range of applications. In 2007, Huang and Zhang [9] introduced the concept of cone metric spaces as a generalization of traditional metric spaces. They proved the existence of a unique fixed point for contractive mappings in complete cone metric spaces. Dajun Guo and V. Lakshmikantham [2] established existence theorems for coupled fixed points for both continuous and discontinuous operators, with applications to initial value problems of ordinary differential equations with discontinuous right-hand sides. Bhaskar and Lakshmikantham [19] further developed the theory by proving the existence of coupled fixed points for mixed monotone mappings in partially ordered metric spaces. In 2008, C. Di Bari [1] presented a common fixed point theorem in cone metric spaces. Later, in 2009 and 2010, I. Altun [6, 7] established several common fixed point theorems in cone metric spaces and ordered cone metric spaces. Additionally, M. Arshad [13] and S. Radenović [16], both in 2009, contributed further by proving common fixed point theorems in cone metric spaces. Recently, Huang and Zhang in [1] generalized the concept of metric spaces by considering vector-valued metrics (cone metrics) with values in an ordered real Banach space. They proved some fixed point theorems in cone metric spaces showing that metric spaces really doesnot provide enough space for the fixed point theory. Indeed, they gave an example of a cone metric space and proved existence of a unique fixed point for a selfmap of which is contractive in the category of cone metric spaces but is not contractive in the category of metric spaces. After that, cone metric spaces have been studied by many other authors (see [1–9] and the references therein). Regarding the concept of coupled fixed point, introduced by Bhaskar and Lakshmikantham [10], we consider the corresponding definition for the mappings on complete cone metric spaces and prove some coupled fixed point theorems in the next section. First, we recall some standard notations and definitions in cone metric spaces. The study of fixed point theory has been a cornerstone in nonlinear analysis due to its wide applications in differential equations, dynamic systems, and optimization. In 2007, **Huang and Zhang** introduced the concept of *cone metric spaces*, which generalize the idea of metric spaces by replacing the real numbers with elements of a cone in a Banach space. This generalization has opened a new avenue for the development of fixed point theory in abstract and generalized settings.

The concept of **coupled fixed points** arises naturally in the study of nonlinear functional equations and systems of equations. A point $(x, y)$is said to be a coupled fixed point of a mapping $F: X \times X \to X$if

$$F(x, y) = x \text{and} F(y, x) = y.$$

This notion was introduced by **Bhaskar and Lakshmikantham (2006)** and has been studied extensively in partially ordered metric spaces and Banach spaces.

Recently, several authors have extended fixed point theorems to cone metric spaces. However, most of these results focus on single-valued mappings or require strong contractive conditions. The objective of this paper is to establish coupled fixed point results under weaker contractive conditions in cone metric spaces

2. PRELIMINARIES

We begin with some basic definitions and notations used throughout the paper.

Definition 2.1 (Cone)

Let E be a real Banach space. A subset $P \subset E$ is called a **cone** if:

1. P is nonempty, closed, and $P \neq \{0\}$;
2. If $a, b \geq 0$ and $x, y \in P$, then $ax + by \in P$;
3. If $x \in P$ and $-x \in P$, then $x = 0$.

The cone P induces a partial order on E defined by:

$$x \leq y \iff y - x \in P.$$

If the interior of P (denoted by $\text{int}(P)$) is nonempty, we write $x \ll y$ whenever $y - x \in \text{int}(P)$.

Definition 2.2 (Cone Metric Space)

Let X be a nonempty set and E a real Banach space with cone P. A function $d: X \times X \rightarrow E$ is called a **cone metric** if for all $x, y, z \in X$:

1. $0 \leq d(x, y)$, and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called a **cone metric space**.

Definition 2.3 (Normal Cone)

A cone P is said to be **normal** if there exists a constant $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \implies \| x \| \leq K \| y \|.$$

3. COUPLED FIXED POINT THEOREMS

Let (X, d) be a cone metric space and E a Banach space with cone P.

Definition 3.1

A mapping $F: X \times X \rightarrow X$ is said to have the **mixed monotone property** if

$$x_1 \leq x_2 \implies F(x_1, y) \leq F(x_2, y)$$

and

$$y_1 \leq y_2 \implies F(x, y_1) \geq F(x, y_2),$$

for all $x, y, x_1, x_2, y_1, y_2 \in X$.

Definition 3.2 (Coupled Fixed Point)

An element $(x, y) \in X \times X$ is called a **coupled fixed point** of F if

$$F(x, y) = x, F(y, x) = y.$$

Theorem 3.1

Let (X, d) be a complete cone metric space with a normal cone P having a nonempty interior, and let $F: X \times X \rightarrow X$ be a continuous mapping satisfying:

$$d(F(x, y), F(u, v)) \leq k\, d(x, u) + l\, d(y, v)$$

for all $x, y, u, v \in X$, where $k, l \geq 0$ and $k + l < 1$.

Then Fhas a unique coupled fixed point $(x^*, y^*) \in X \times X$.

Proof:
Let $x_0, y_0 \in X$be arbitrary. Define sequences $\{x_n\}$and $\{y_n\}$by:

$$x_{n+1} = F(x_n, y_n), y_{n+1} = F(y_n, x_n), n = 0,1,2, \dots$$

We show that $\{x_n\}$and $\{y_n\}$are Cauchy sequences.
Consider:

$$d(x_{n+1}, x_n) = d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \le k\, d(x_n, x_{n-1}) + l\, d(y_n, y_{n-1}).$$

Similarly,

$$d(y_{n+1}, y_n) = d(F(y_n, x_n), F(y_{n-1}, x_{n-1})) \le k\, d(y_n, y_{n-1}) + l\, d(x_n, x_{n-1}).$$

Adding these inequalities gives:

$$d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \le (k + l)[d(x_n, x_{n-1}) + d(y_n, y_{n-1})].$$

Iterating, we obtain:

$$d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \le (k + l)^n [d(x_1, x_0) + d(y_1, y_0)].$$

Since $k + l < 1$, the right-hand side tends to 0 as $n \to \infty$. Thus, both sequences are Cauchy.
Since $(X, d)$is complete, there exist $x^*, y^* \in X$such that:

$$x_n \to x^*, y_n \to y^*.$$

Using the continuity of F,

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F(x^*, y^*),$$

and

$$y^* = F(y^*, x^*).$$

Hence, $(x^*, y^*)$is a coupled fixed point.
To prove uniqueness, assume $(x', y')$is another coupled fixed point. Then:

$$d(x^*, x') = d(F(x^*, y^*), F(x', y')) \le k\, d(x^*, x') + l\, d(y^*, y').$$

Similarly,

$$d(y^*, y') \le k\, d(y^*, y') + l\, d(x^*, x').$$

Adding and rearranging gives:

$$(1 - k - l)[d(x^*, x') + d(y^*, y')] \le 0,$$

which implies $x^* = x'$and $y^* = y'$.
Hence, the coupled fixed point is unique.

4. COROLLARIES AND REMARKS

Corollary 4.1
If $F(x, y) = F(y, x)$for all $x, y \in X$, then the coupled fixed point theorem reduces to a single fixed point theorem in cone metric spaces.

Remark 4.1

If $E = \mathbb{R}$and $P = [0, \infty)$, then the cone metric $d$becomes an ordinary metric, and Theorem 3.1 reduces to the classical **Banach contraction principle**.

5. EXAMPLE

Let $E = \mathbb{R}^2$with the usual norm and $P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$. Define $X = [0,1]$and

$$d(x, y) = (|\, x - y \,|, \alpha \,|\, x - y \,|),$$

where $\alpha > 0$.
Let $F : X \times X \to X$be defined by

$$F(x, y) = \frac{x + y}{4}.$$

Then for any $x, y, u, v \in X$,

$$|\, F(x, y) - F(u, v) \,| = \frac{1}{4}(|\, x - u \,| + |\, y - v \,|) \leq \frac{1}{4}[|\, x - u \,| + |\, y - v \,|].$$

Hence, the contractive condition holds with $k = l = \frac{1}{4}$, and $k + l = \frac{1}{2} < 1$.
Thus, by Theorem 3.1, Fhas a unique coupled fixed point $(x^*, y^*) = (0,0)$.

6. CONCLUSION

We have established coupled fixed point theorems for mappings satisfying contractive conditions in cone metric spaces. The results generalize and extend classical fixed point theorems by incorporating the structure of cones in Banach spaces. Future work can focus on extending these results to **cone b-metric spaces**, **intuitionistic fuzzy cone spaces**, or **partial cone normed linear spaces**.

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